Silo is a Finite Game

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1 Introduction

Silo[1] is an abstract strategy board game by Mark Steere. In the game, two players take turns moving pieces along a finite one-dimensional board of discrete spaces. There may be multiple pieces on a single space, which are arranged in a stack. Each player has an assigned direction and an assigned set of pieces. Each of the finitely many pieces is assigned to exactly one of the two players. On a player's turn, they choose one of their own pieces and move it one space in their assigned direction, placing it on top of the stack of pieces in the destination space. The piece that the player moves must be the highest of their own pieces in that stack. If any of their opponent's pieces are on top of the moved piece, the opponent's pieces remain on top of the moved piece and join the destination stack. If a player cannot move because all of their pieces are on the space farthest in their assigned direction, then the player loses their turn. Players may not pass their turns voluntarily. If all of a player's pieces are arranged consecutively in the same stack, then that player wins and the game ends immediately

Steere does not insist upon any particular board size or number of pieces. He suggests boards of even length with same-sized stacks of pieces on each space, each stack belonging entirely to the same player, alternating ownership along the board, and the end spaces have pieces belonging to the player moving away from that end.

2 Finiteness

Steere[2] makes a point of designing games that terminate in finite time without the possibility of draws. This property is clear in many of the games he has designed because they involve the irreversible capture or advancement of pieces. Silo, on the other hand, does not involve piece capture, and progress may be reversed when a piece "rides" the piece of an opponent. It will be established below that Silo is a finite game, but this fact is not entirely obvious.

[1] Rules of Silo: https://www.marksteeregames.com/Silo_rules.pdf

[2] Steere's website describing other games: https://www.marksteeregames.com/

2.1 Notation

To prove the main result, it will be useful to define some notation. Say the board directions are left and right and the players have corresponding pieces represented by the symbols $-$ and $+$. Represent empty space with the symbol o.

In a game with m total pieces and n board spaces, represent board positions as $m \times n$ matrices of these symbols. It will be convenient to index the columns of these matrices from bottom to top, so the bottom row of the state matrix P (representing the pieces or blank spaces in contact with the board) is written as $[P_{1,1}, P_{1,2}, \ldots, P_{1,n}]$. For example, a length-4 board with two left-moving pieces topped by a right-moving piece on the third space and one right-moving piece in the leftmost space would be represented as

$$P = \begin{bmatrix} o & o & o & o \\ o & o & + & o \\ o & o & - & o \\ + & o & - & o \end{bmatrix}.$$

In this example, $P_{1,1} = P_{3,3} = +$ and $P_{1,3} = P_{2,3} = -$ and all other entries are o.

Now define an order on the board states by ordering the symbols

$$- < o < +$$

and using column-major lexicographical order on the board states: the matrices are ordered by considering the leftmost differing column of the matrices with the bottommost differing entry of that column giving the ordering of the matrices.

More precisely, let P and Q be any two board states. If $P_{i,j} = Q_{i,j}$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$, then $P = Q$. Otherwise, let j^* be the smallest column index where P and Q differ. In other words, j^* is the smallest j such that, for at least one row i, $P_{i,j} \neq Q_{i,j}$. Now, let i^* be the smallest row index such that $P_{i^*,j^*} \neq Q_{i^*,j^*}$. If $P_{i^*,j^*} < Q_{i^*,j^*}$, then $P < Q$. Otherwise, $Q < P$.

Call i^*, j^* the "key" for comparing P and Q.

For example

$$\begin{bmatrix} o & o & o & o \\ o & o & o & o \\ o & - & o & - \\ \color{red}{o} & + & o & - \end{bmatrix} < \begin{bmatrix} o & o & o & o \\ o & o & + & o \\ o & o & - & o \\ \color{red}{+} & o & - & o \end{bmatrix}$$

because the 1,1 entries (colored red) of these matrices are o and $+$. If the 1,1 entries of two matrices are different, then 1,1 is their key because there can be no entry farther left or down than 1,1. As another example, the key of these matrices is 2,3:

$$\begin{bmatrix} o & o & o & o \\ o & o & + & o \\ o & o & \color{red}{-} & o \\ o & o & - & + \end{bmatrix} < \begin{bmatrix} o & o & + & o \\ o & o & - & o \\ o & o & \color{red}{+} & o \\ o & o & - & o \end{bmatrix}.$$

2.2 Results

Lemma 1. With the possible exception of the initial state, no reachable game state has a winning position for both players at once.

Proof. A player's move cannot bring into contact any of their opponents pieces that were not already together. Therefore, a single move creates the winning condition for at most one player. □

Lemma 2. In every reachable game state, at least one player is able to move a piece.

Proof. Suppose, for contradiction, that neither player can take a turn. Then all pieces are on their most advanced space, and each player has all of their pieces in a single stack. Each player's pieces must be stacked consecutively because their opponent's pieces are on the other end of the board. In other words, it is only possible for both players to have no moves if both players have their pieces in a winning configuration. However, both players cannot reach a winning state simultaneously (Lemma 1), and one of the players must have reached a winning configuration first, at which point the game should have ended, which is a contradiction. Thus, it is not possible for both players to have no moves. □

Lemma 3. If a player moves a piece legally and transfers the game state from P to Q, then $Q < P$.

Proof. Consider a move of the left-moving player. Say that the player moves a piece from column $j + 1$ to column $j < n$. The only columns of P and Q that differ are columns j and $j + 1$, so column j is the leftmost changed column. In the order definition's notation, $j^* = j$.

Suppose that the height of the stack in column j of state P was $i - 1$, i.e., $P_{i-1,j} \neq o$ is not empty and $P_{i,j} = o$ is empty. The stack of pieces in column j of P remain in place during the move, so $P_{k,j} = Q_{k,j}$ for $k < i$. The left-moving piece that was moved from column $j+1$ to column j lands on top of the existing stack in column j and therefore ends in position i, j. Thus, $Q_{i,j} = -$ while $P_{i,j} = o$, so entry i is the bottommost entry in the column to change. In the order definition's notation, $i^* = i$. Consequently, i, j is the key for P and Q, and since the i, j entries of the matrices are respectively o and $-$, we have shown that $Q < P$.

The case of the right-moving player is similar, but the key is the position occupied by the player's piece in the *first* state P. When the player moves their piece to the right, the entry corresponding to this moved piece changes from $+$ to o, decreasing the board state. □

Here's an example of Lemma 3 in action. Say that left-moving player moves the piece at 1,3 to transition from P to Q given below:

$$P = \begin{bmatrix} o & o & o & o \\ o & o & o & o \\ o & o & + & o \\ - & + & - & o \end{bmatrix}, \quad Q = \begin{bmatrix} o & o & o & o \\ o & + & o & o \\ o & - & o & o \\ - & + & o & o \end{bmatrix}.$$

The first columns of P and Q are identical, but the second column was the destination of the player's move, so the second columns are the leftmost differing columns ($j^* = 2$). Column 2 of P had one piece in it, so the moving player's piece lands at position 2,2. Entry 2,1 (the piece that was landed upon) did not change, so the bottommost changed entry is entry 2 ($i^* = 2$). Finally, since $Q_{i,j} = - < o = P_{i,j}$, we see that $Q < P$.

Theorem 4. *Silo is a finite game.*

Proof. Each game state is defined by two parts:

1. designation of the player on move, and

2. an arrangement of finitely many pieces into finitely many stacks.

These components can only be combined in finitely many ways, so the space of game states is finite. In particular, per the notation given, the number of possible states of a length-n board with m pieces is no more than the number of $m \times n$ matrices with each entry being a member of $\{-, o, +\}$. There are 2^{mn} such matrices and fewer possible board states. The possibility of a different player being on move for the same board state contributes a factor of two to the number of possible game states, but this does not affect finiteness.

Lemma 2 shows that the game states may not enter a deadlocked cycle of lost turns, therefore players will continue to move pieces unless the game ends. Lemma 3 shows that moves cannot transition though a cycle since the sequence of board states is strictly decreasing. Thus, the game progresses acyclically though a finite set of game states and must terminate in finitely many moves. □